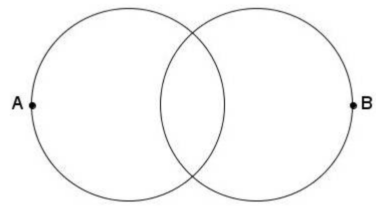
4th Annual Lexington Mathematical Tournament
Individual Round

Solutions

1. **Answer**: $\boxed{780}$ A common multiple of 20, 12 and 13 must have factors of 2^2, 3, 5, and 13. Therefore, the least common multiple is $2^2 \cdot 3 \cdot 5 \cdot 13 = 780$.

2. **Answer**: $\boxed{17}$ It is easy to see that the greatest distance between A and B is achieved by placing A and B on opposite ends of different circles, as shown. Therefore, the maximum distance is $5 + 7 + 5 = 17$.



3. **Answer**: $\boxed{0}$ No matter how the integers are placed, 1 and 2 must be in the same row, column, or main diagonal. Thus, the common sum must be 3. However, 3 and 4 will also be in the same row, column, or main diagonal, so the common sum must be 7. The common sum clearly cannot be multiple different values, so there are 0 possible arrangements.

4. **Answer**: $\boxed{\frac{2}{5}}$ There are 4 white socks and 2 red socks in total. To pick 2 socks, there are 6 ways to pick the first sock and then 5 ways to pick the second. To pick 2 white socks, there are 4 ways to pick the first sock and then 3 ways to pick the second. The probability he picked two white socks is $\frac{4 \cdot 3}{6 \cdot 5} = \frac{2}{5}$.

5. **Answer**: $\boxed{22}$ The maximum sum of the time is at 9:59 because 9 gives the greatest sum for any hour and 59 gives the greatest sum for any minute. The minimum sum of the time is obviously 1, which can be achieved at 10:00 or 1:00. Therefore, the maximum positive difference between the sum at one time and the sum one minute later occurs from 9:59 to 10:00. The sum of 9:59 is 23 and the sum of 10:00 is 1, so the difference is $23 - 1 = 22$.

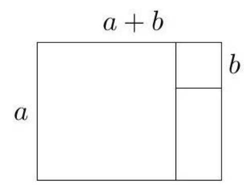
6. **Answer**: $\boxed{6}$ After placing parentheses, the expression will evaluate to p/q, where p is the product of some of the numbers $\{1, 2, 3, 4\}$ and q is the product of the remaining numbers. Furthermore, it is clear that 1 is part of the product p and that at least one number is in q, so $q > 1$. Thus, to maximize p and minimize q, we attempt to find a solution where $q = 2$ and $p = 12$. This is achievable by placing parentheses around $2 \div 3 \div 4$, so the answer is $12/2 = 6$.

7. **Answer**: $\boxed{\frac{1}{4}}$ Since 3/5 of the people of the room before the cosmologists arrive are astronomers, suppose there are $3x$ astronomers and $2x$ astrophysicists. In addition, suppose there are y cosmologists. When they arrive, there are $5x + y$ people in total. We are given that after the cosmologists arrive, 3/10 of the scientists in the room are astrophysicists, so $\frac{2x}{5x+y} = \frac{3}{10}$. Cross multiplying, we get $20x = 15x + 3y \Rightarrow 5x = 3y \Rightarrow x = \frac{3y}{5}$. The fraction of cosmologists in the room is $\frac{y}{5x+y} = \frac{y}{4y} = \frac{1}{4}$.

8. **Answer**: $\boxed{55}$ In order for the minuteman to walk halfway down, or 600 steps, he must take $600/40 = 15$ minutes. It takes the elevator 40 minutes to move him back up 600 steps, so the minuteman must be moving up at a net speed 15 steps per minute. If he is walking down at 40 steps per minute, the elevator must be moving up at $15 + 40 = 55$ steps per minute.

9. **Answer**: $\boxed{57}$ Rearranging the expression, we have $x_n + x_{n+2} = x_{n-1} + x_{n-3}$. Thus, it follows that
$$x_1 + x_3 = x_2 + x_4 = x_3 + x_5 = \cdots,$$
and in particular, $x_1 + x_3 = x_3 + x_5$, so $x_1 = x_5$. Continuing, we deduce that $x_{4a+1} = x_{4b+1}$ for all nonnegative integers a and b. We then get $x_{2013} = x_1 = 57$.

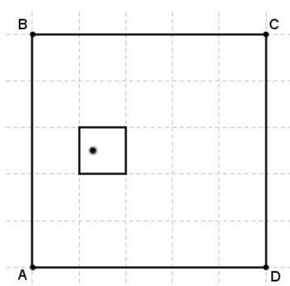
10. **Answer**: $\boxed{12}$ Call the length of the larger square a and the length of the smaller square b. The area of the small square is b^2. The side lengths of the rectangle are a and $a+b$, so the area of the rectangle is $a^2 + ab$. The length of the diagonal of the rectangle can be found using the Pythagorean Theorem and is equal to $\sqrt{(a^2) + (a+b)^2} = \sqrt{2(a^2+ab)+b^2}$. Substituting known values, the length of the diagonal equals $\sqrt{2(60)+24} = \sqrt{144} = 12$.



11. **Answer**: $\boxed{140}$ There are 2 ways to choose the front adult, and then the last adult is fixed. There is only one possible sequence to line up the girls and one possible sequence to line up the boys, as they must be in alphabetical order. For a line of 8 people, we pick 4 of them to be the boys, and the other 4 are the girls. Thus, there are $\binom{8}{4} = 70$ ways to line up the girls and boys. The total number of possible lines is $2 \cdot 70 = 140$.

12. **Answer**: $\boxed{\left(\frac{5}{4}, \frac{5}{2}\right)}$ The y-coordinate must be half of the height of Flatland since the map is centered at half the height, so the y-coordinate is $5/2$.

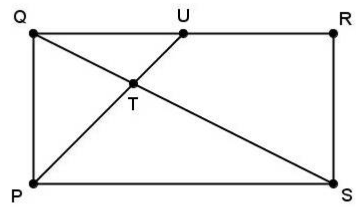
To find the x-coordinate, we note that the ratio of the distance from the point to the left edge of the land to the width of the land must be equal to the ratio of the distance from the point to the left edge of the map to the width of the map. Call the x-coordinate of the point a. We set up the ratio $\frac{a}{5} = \frac{a-1}{1}$. Then $a = 5a - 5$, so $5 = 4a \Rightarrow a = 5/4$.



The coordinates of the point are $\left(\frac{5}{4}, \frac{5}{2}\right)$.

13. **Answer**: $\boxed{15}$ Each number x appears x times, so each number x contributes x^2 to the sum. The formula for the sum of the squares from 1 to x is $\frac{(x)(x+1)(2x+1)}{6}$. The number of elements is the sum of the numbers from 1 to x, which is $\frac{x(x+1)}{2}$. The average value is therefore $\frac{\frac{(x)(x+1)(2x+1)}{6}}{\frac{x(x+1)}{2}} = \frac{2x+1}{3}$. Plugging in $x = 22$, we get $\frac{45}{3} = 15$.

14. **Answer**: $\boxed{\frac{3}{8}}$ Let $[\mathcal{F}]$ denote the area of figure $\mathcal{F}$. Since $PT : TU = 3 : 1$, $ST : TQ = 3 : 1$. Since triangles PTS and TPQ share the same height, $[PTS] : [TPQ] = 3 : 1$ and $[PTS] : [PQS] = 3 : 4$. Since triangle PQS is half of the rectangle, $[PTS] : [PQRS] = \frac{\frac{3}{4}}{2} = \frac{3}{8}$.



15. **Answer**: $\boxed{4, \frac{9}{2}}$ Plugging $x = A$ and $x = B$ in, we have $A^3 + AB + C = AB^2 + B^2 + C \Rightarrow A^3 - AB^2 - B^2 + AB = 0$. Factoring gives us $A(A + B)(A - B) + B(A - B) = 0 \Rightarrow (A^2 + AB + B)(A - B) = 0$. Thus, either $A^2 + AB + B = 0$ or $A - B = 0$. However, $A \neq B$ since $A + 6 = B$, so $A - B \neq 0$. Substituting $A + 6 = B$ into $A^2 + AB + B = 0$, we have $2A^2 + 7A + 6 = 0$ which factors to $(2A + 3)(A + 2) = 0$. Therefore, $A = \frac{-3}{2}$ or -2 and $B = \frac{9}{2}$ or 4.

Alternate solution: The quadratic $f(x)$ achieves its minimum or maximum at $x = -\frac{B}{2A}$. The two solutions are A and B, so $f(x)$ also achieves its minimum or maximum at $x = \frac{A+B}{2}$. Thus,

$$\frac{A+B}{2} = -\frac{B}{2A} \Rightarrow A^2 + AB + B = 0.$$

We finish as we did above.

16. **Answer**: $\boxed{\frac{42}{55}}$ The prime factors of 98 are 2 and 7. We first calculate the sum of the integers not relatively prime to 98. We find the sum of the integers with a factor of 2:

$$2 + 2 \cdot 2 + 2 \cdot 3 + \cdots + 2 \cdot 48 = 2(1 + 2 + 3 + \cdots + 48) = 48 \cdot 49.$$

Now, we find the sum of the integers with a factor of 7:

$$7 + 7 \cdot 1 + 7 \cdot 2 + \cdots + 7 \cdot 13 = 7(1 + 2 + \cdots + 13) = \frac{7 \cdot 13 \cdot 14}{2} = 49 \cdot 13.$$

Of course, we counted the integers with a factor of 14 twice. These have a sum of

$$14 + 14 \cdot 2 + \cdots + 14 \cdot 6 = 14(1 + 2 + \cdots + 6) = \frac{14 \cdot 6 \cdot 7}{2} = 49 \cdot 6.$$

In order to only count this once, we must subtract the sum once. The overall sum is thus $49 \cdot 48 + 49 \cdot 13 - 49 \cdot 6 = 49 \cdot 55$. The sum of the integers relatively prime to 98 and the sum of the integers not relatively prime to 98 must add up to the sum of the integers less than 98, which is $\frac{97 \cdot 98}{2} = 49 \cdot 97$. The sum of integers relatively prime to 98 is $49 \cdot 97 - 49 \cdot 55 = 49 \cdot 42$. The answer is thus $\frac{49 \cdot 42}{49 \cdot 55} = \frac{42}{55}$.

Alternate solution: If x is relatively prime to 98, then $98 - x$ is also relatively prime to 98. Similarly, if x is not relatively prime to 98, then $98 - x$ is not relatively prime to 98. Thus, the average value of the integers that make up α is 49, as is the average value of the integers that make up β, so the ratio of the sums is the ratio of the numbers of terms in the sums. There are $\varphi(98) = 42$ numbers relatively prime to 98 less than 98, so there are 42 terms in α and $97 - 42 = 55$ terms in β, so $\alpha/\beta = 42/55$.

17. **Answer**: $\boxed{\frac{9}{8}}$ Call the sum S. Then,

$$\frac{S}{3} = \frac{1}{9} + \frac{3}{27} + \frac{6}{81} + \cdots .$$

Subtracting this from S, we have

$$\frac{2S}{3} = \frac{1}{3} + \frac{2}{9} + \frac{3}{27} + \frac{4}{81} + \cdots .$$

Dividing by 3 again, we have

$$\frac{2S}{9} = \frac{1}{9} + \frac{2}{27} + \frac{3}{81} + \cdots .$$

Subtracting this from $\frac{2S}{3}$, we get

$$\frac{4S}{9} = \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \cdots .$$

This is just a geometric series with common ratio 1/3 and first term 1/3, so the sum is $\frac{4S}{9} = \frac{\frac{1}{3}}{1-\frac{1}{3}} = \frac{1}{2}$. Therefore, $S = \frac{9}{8}$.

18. **Answer**: $\boxed{504}$ Denote N, E, S, W to be a step by the bug in the north, east, south and west direction respectively. To get from (0,0) to (2,2) the bug moves a net 2N and 2E. In between, the bug can move another 2N and 2S, 2E and 2W, or 1 more in each direction.

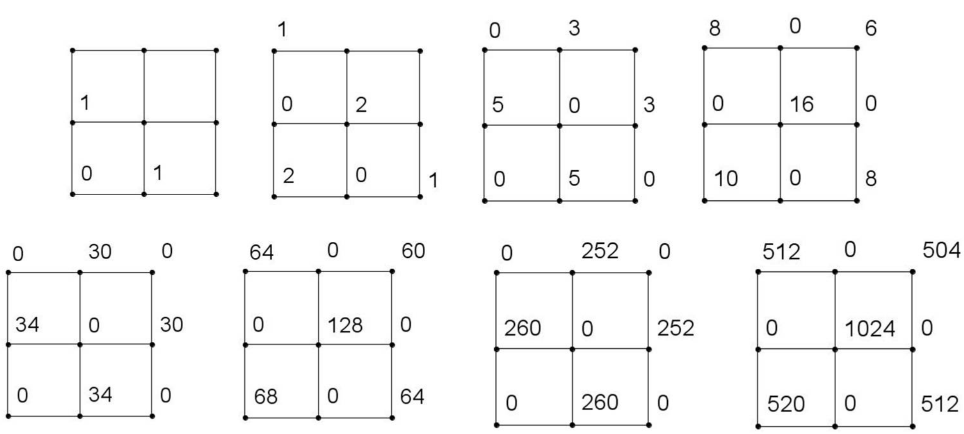
Case 1: The bug moves another 2N and 2S. The bug can either move NNSNSN, NNSSNN, NSNNSN, or NSNSNN. (Note that it cannot move NNN first since then it would be at a location where the y-coordinate is 3.) There are 8 steps, and 2 of them must be E, so there are $4\binom{8}{2} = 112$ paths.

Case 2: The bug moves another 2E and 2W. This is the exact same as Case 1 but with the orientation changed, so there are 112 paths.

Case 3: The moves another N, S, E, and W. Considering only the N, S directions, there are 2 ways for the bug to move: NNSN or NSNN. Similarly there are 2 ways for the bug to move in the E, W directions. Out of the 8 steps, 4 steps must be in the N, S directions, so there are a total of $2 \cdot 2 \cdot \binom{8}{4} = 280$ ways.

There are a total of $112 + 112 + 280 = 504$ ways.

Alternate solution: We draw a sequence of 8 grids of available points, and at grid n for $n = 1, 2, \ldots, 8$, we label each of the nine points in the grid with the number of paths the bug can take to be at that point in n seconds. Thus, the label for a point in one grid is equal to the sum of the labels of its neighbors in the previous grid. The work is shown below. Maybe you will notice some interesting patterns!



19. **Answer**: $\boxed{101}$ The maximum possible value of $f(n)$, where n is less than 2013, is 28. Since $f(f(n))$ and $f(f(f(n)))$ must be different, $f(f(n)) \geq 10$. However, since the maximum value of $f(n)$ is 28, the maximum value of $f(f(n))$ is 10. Therefore, we wish to count the values of n such that $f(f(n)) = 10$, so $f(n)$ can be 19 or 28.

Case 1: $f(n) = 19$:

Case 1.1: n is a 3-digit number abc. Then, $a + b + c = 19$. Let $a = 9 - x$, $b = 9 - y$, and $c = 9 - z$. Then, $27 - x - y - z = 19$, or $x + y + z = 8$. We divide 8 units into 3 groups, which is the equivalent of taking 10 units and using 2 as dividers. Since the maximum of x, y, or z is 8, all possible combinations work. (Note that if we divided 19 units into 3 groups, we will have cases where a, b, or c is greater than 9.) There are $\binom{10}{2} = 45$ ways.

Case 1.2: n is a 4 digit number. The thousands digit must be 1 since the largest value of f(n) with 2 in the thousands place is if $n = 2009$, which would produce $f(n) = 11$. Thus, let our 4 digit number be $1abc$, and with similar reasoning as before, let $a = 9 - x$, $b = 9 - y$, and $c = 9 - z$. Then, $28 - x - y - z = 19$, or $x + y + z = 9$. We divide 9 units into 3 groups, which is the equivalent of taking 11 units and using 2 as dividers. Since the maximum of x, y, or z is 9, and a, b, or c may be 0, all combinations work. There are $\binom{11}{2} = 55$ ways.

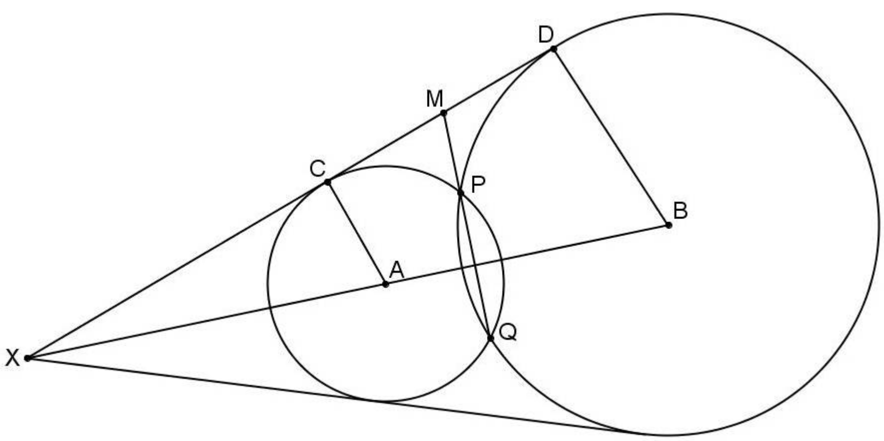
Case 2: $f(n) = 28$: This only occurs when $n = 1999$. There is only 1 way.

The total number of numbers is $45 + 55 + 1 = 101$.

20. **Answer**: $\boxed{8\sqrt{3} - 12}$ Let A and B be the centers of ω_1 and ω_2 respectively. Let $Q \neq P$ be the other point of intersection of the two circles. We claim that M, P, and Q all lie on the same line. Suppose that they do not, and suppose the line through M and P intersects ω_1 at A_1 and ω_2 at A_2. Thus, $A_1 \neq A_2$. By repeated application of Power of a Point,

$$(MC)^2 = (MP)(MA_1) \neq (MP)(MA_2) = (MD)^2,$$

but since $MC = MD$ by the definition of M, we have a contradiction, so our assumption that M, P, and Q did not all lie on a line was false.



Using Power of a Point, we get

$$(MC)^2 = (MP)(MQ) = (MP)(MP + PQ),$$

so it suffices to find PQ and MC. To find PQ, we note that PAB is a 13-14-15 triangle with area 84. (To verify this, we can use Heron's formula or note that a 13-14-15 triangle can be composed from joining two right triangles, one with side lengths 5-12-13 and the other with side lengths 9-12-15.) Since $AB = 14$, it follows that the height from P to $\overline{AB}$ is 12. By symmetry, it follows that $PQ = 24$.

To find MC, let the line passing through A and B and the line passing through C and D intersect at X. Triangles XCA and XDB are similar since $\overline{CA}$ and $\overline{DB}$ are perpendicular to $\overline{CD}$. If $XA = a$, then $\frac{a}{CA} = a + \frac{14}{DB} \Rightarrow \frac{a}{13} = a + \frac{14}{15}$. Cross multiplying and simplifying, we have $a = 7 \cdot 13 = 91$. Thus, $XC = \sqrt{49 \cdot 13^2 - 13^2} = 13\sqrt{48} = 52\sqrt{3}$. Since XC and CA are in the ratio $4\sqrt{3} : 1$, XD and DB are in the same ratio. Thus, XD is $60\sqrt{3}$ and $CD = 8\sqrt{3}$. Point M is the midpoint of CD, so $CM = 4\sqrt{3}$.

Substituting these values back in, we get $48 = (MP)(MP + 24)$. Expanding and rearranging, we have the quadratic $(MP)^2 + 24(MP) - 48 = 0$. Using the quadratic formula, we have $MP = \pm 8\sqrt{3} - 12$, but obviously MP cannot be negative, so it is equal to $8\sqrt{3} - 12$.

Note: The set of all points having equal power with respect to each of two given circles is called the *radical axis* of the two circles. This set is a line, and when the circles have two intersection points, the line passes throughout these two points because their powers are zero.